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COMMENT ON FIVE-DIMENSIONAL GEODESY

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Recently several authors have studied the Induced Matter Theory (IMT).[1,7,9] The IMT is based on the Kaluza–Klein idea[3,4] and postulates that the vacuum five-dimensional Einstein field equations give rise to a four-dimensional theory with matter sources, and hence gives a prescription for a possible geometrical origin for matter. The IMT has recently received some theoretical support in that it was proven that any analytic n-dimensional Riemannian space can be locally embedded in an $(n+1)$-dimensional Ricci-flat space,[6] so that all general relativistic space–times can be locally embedded in a five-dimensional Ricci-flat space–time.

In the IMT, four-dimensional space–time is locally and isometrically embedded in a five-dimensional vacuum space–time. Writing

$$ds^2 = g_{ab}\, dx^a\, dx^b = g_{\alpha\beta}\, dx^\alpha\, dx^\beta + \phi^2\, d\eta^2 \tag{1}$$

$(a, b = 0,1,2,3,4;\ \alpha,\beta = 0,1,2,3;\ \eta = x^4)$, the five-dimensional vacuum field equations are

$${}^{(5)}R_{ab} = 0\,. \tag{2}$$

The equations can then be written as

$${}^{(4)}R_{\alpha\beta} = \phi^{-1}\phi_{;\alpha\beta} - \frac{1}{2}\,\phi^{-2}\left\{\phi^{-1}\overset{*}{\phi}\overset{*}{g}_{\alpha\beta} - \overset{**}{g}_{\alpha\beta} + g^{\lambda\mu}\overset{*}{g}_{\alpha\lambda}\overset{*}{g}_{\beta\mu} - \frac{1}{2}\,g^{\mu\nu}\overset{*}{g}_{\mu\nu}\overset{*}{g}_{\alpha\beta}\right\}, \tag{3}$$

where ${}^{(4)}R_{\alpha\beta}$ is the four-dimensional Ricci tensor constructed from $g_{\alpha\beta}$ and "$*$" denotes differentiation with respect to η. Hence we have that general relativity is embedded in the hypersurface Σ_4 where $\eta = \eta_0 =$ constant with metric $g_{\alpha\beta}$ and energy momentum tensor $T_{\alpha\beta}$ defined by

$$T_{\alpha\beta} = {}^{(4)}R_{\alpha\beta} - \frac{1}{2}\,{}^{(4)}R g_{\alpha\beta}\,. \tag{4}$$

[The equations ${}^{(5)}R_{4\alpha} = 0 = {}^{(5)}R_{44}$ represent constraints (on, for example, ϕ)]. Consequently, the matter content of the four-dimensional universe is geometrical in nature.

In addition, it has been postulated[2] that freely-falling test particles follow geodesics in the five-dimensional (vacuum) space–time. This postulate is an additional assumption in IMT and is different in nature from the other postulates described above. Indeed, IMT is a self-consistent theory independent of this additional assumption.

Moreover, the (four-dimensional) Bianchi identities are automatically satisfied in the intrinsic four-dimensional hypersurfaces Σ_4, and consequently the (four-dimensional) energy momentum tensor is conserved, i.e.

$$T_{\alpha\beta;\gamma}g^{\beta\gamma} = 0\,, \tag{5}$$

where the semicolon denotes covariant differentiation with respect to $g_{\alpha\beta}$. Hence, the motion of the matter in the hypersurfaces Σ_4 is constrained by (5). In particular, if the four-dimensional matter is (pressure-free) dust, then Eq. (5) implies that the dust particles follow (four-dimensional) geodesics in Σ_4. Presumably, if the additional five-dimensional geodesy assumption is to be consistent, in the case of dust these four-dimensional geodesics must be related to the geodesics in five dimensions (at least on Σ_4).

To examine the assumption of five-dimensional geodesy, let us consider the cosmological solutions of Ponce de Leon[5] in which the metric is given by

$$ds^2 = -\eta^2\,dt^2 + t^{2/\alpha}\eta^{2/(1-\alpha)}(dx^2+dy^2+dz^2) + \frac{\alpha^2}{(1-\alpha)^2}\,t^2\,d\eta^2\,, \tag{6}$$

where the parameter $\alpha > 0\ (\neq 1)$. In the IMT, (6) describes a class of perfect fluids in the hypersurfaces Σ_4 $(\eta = \eta_0)$ with the equation of state $p = \mu(2\alpha-3)/3$, where p is the pressure and μ is the energy density of the fluid. Clearly, when $\alpha = 3/2$ this is the equation of state for dust $(p = 0)$.

Based on (6), the five-dimensional geodesic equations read

$$\ddot{x} = -2\left(\frac{\dot{t}}{\alpha t} + \frac{\dot{\eta}}{(1-\alpha)\eta}\right)\dot{x}\,, \qquad \text{(similarly for } y \text{ and } z\text{)} \tag{7}$$

$$\ddot{t} = -\frac{1}{\alpha}\,\frac{t^{2/\alpha}\eta^{2/(1-\alpha)}}{\eta^2}\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right) - 2\,\frac{\dot{t}\dot{\eta}}{\eta} - \frac{\alpha^2}{(1-\alpha)^2}\,\frac{t}{\eta^2}\,\dot{\eta}^2\,, \tag{8}$$

$$\ddot{\eta} = \frac{(1-\alpha)}{\alpha}\,\frac{t^{2/\alpha}\eta^{2/(1-\alpha)}}{t^2}\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right) - 2\,\frac{\dot{t}\dot{\eta}}{t} - \frac{(1-\alpha)^2}{\alpha^2}\,\frac{\eta}{t^2}\,\dot{t}^2\,, \tag{9}$$

where "$\cdot$" $\equiv d/ds$ denotes differentiation with respect to the five-dimensional affine parameter. In order for a particle to remain on an $\eta = \eta_0$ hypersurface (Σ_4), $\dot{\eta} = \ddot{\eta} = 0$ is required on Σ_4. Using this in (9), one obtains

$$\dot{t}^2 = \frac{1}{(1-\alpha)}\,t^{2/\alpha}\eta_0^{2\alpha/(1-\alpha)}(\dot{x}^2+\dot{y}^2+\dot{z}^2)\,. \tag{10}$$

However, this expression does not satisfy (8) (using (7)). Therefore, should "test" particles travel along five-dimensional geodesics, they cannot remain on the hypersurface $\eta = \eta_0$ and consequently they cannot travel along the four-dimensional geodesic curves.

As an example to further illustrate this lack of four-dimensional geodesy, let us examine the dust solution ($p = 0$; i.e. $\alpha = 3/2$), where Eq. (5) indicates that the (four-dimensional) fluid velocities are geodesic. Let us then investigate whether the five-dimensional geodesic equations can reduce to the four-dimensional geodesic equations by expressing the four-dimensional components of the five-dimensional geodesic equations,

$$\frac{d^{(5)}u^{\alpha}}{ds} + {}^{(5)}\Gamma^{\alpha}_{bc}\,{}^{(5)}u^{b}\,{}^{(5)}u^{c} = 0\,, \tag{11}$$

(where ${}^{(5)}u^a \equiv dx^a/ds$) in terms of their four-dimensional counterparts[10]

$$\frac{d^{(4)}u^{\alpha}}{d\lambda} + {}^{(4)}\Gamma^{\alpha}_{\beta\gamma}\,{}^{(4)}u^{\beta}\,{}^{(4)}u^{\gamma} = \frac{-B^2}{(1+B^2/\phi^2)\phi^3}\left[\phi^{;\alpha} + \left(\frac{\phi}{B}\frac{dB}{d\lambda} - \frac{d\phi}{d\lambda}\right){}^{(4)}u^{\alpha}\right] - g^{\alpha\beta}g^{*}_{\beta\gamma}\,{}^{(4)}u^{\gamma}\,\frac{d\eta}{d\lambda}\,, \tag{12}$$

where $B \equiv -\phi^2 d\eta/d\lambda$, ${}^{(4)}u^{\alpha} \equiv dx^{\alpha}/d\lambda$, and λ is the four-dimensional affine parameter ($d\lambda^2 = g_{\alpha\beta}\,dx^{\alpha}\,dx^{\beta}$). If ${}^{(4)}u^{\alpha}$ are geodesic, then the right-hand side of (12) vanishes.

Using the velocities (see Ref. 8 with $\alpha = 3/2$)

$${}^{(5)}u^0 = \frac{\mp 3}{2\sqrt{2}\eta}\,, \qquad {}^{(5)}u^l = 0\ (l = 1\text{–}3)\,, \qquad {}^{(5)}u^4 = \frac{\pm 1}{6\sqrt{2}t}\,, \tag{13}$$

which satisfy (11), we find that (12) becomes

$$\frac{d^{(4)}u^0}{d\lambda} = \frac{1}{9t\eta^2}\,, \tag{14}$$

$$\frac{d^{(4)}u^l}{d\lambda} = 0 \qquad (l = 1\text{–}3)\,. \tag{15}$$

Furthermore, the coordinates t and η can be explicitly expressed in terms of λ:

$$t = \left[\mp\frac{8}{9C}\,\lambda + k\right]^{9/8}\,, \tag{16}$$

$$\eta = C\left[\mp\frac{8}{9C}\,\lambda + k\right]^{-1/8}\,, \tag{17}$$

(C and k are integration constants).

Concluding Remarks

If the four-dimensional velocities are geodesic, then $t \propto \lambda$. Both Eqs. (14) and (16) suggest that dust particles following a five-dimensional geodesic *cannot* follow four-dimensional geodesics. However, in a sense, the right-hand side of (14) becomes negligible at "late times".[8] In addition, it is apparent that particles following a five-dimensional geodesic cannot remain on hypersurfaces $\eta = \eta_0$, as demonstrated from Eqs. (7)–(10) and (17) in the case of dust. Therefore, it would seem that the five-dimensional geodesy postulate in the formalism of IMT needs further consideration.

References

1. A. A. Coley, *Ap. J.* **427**, 585 (1994).
2. D. Kalligas, P. S. Wesson and C. W. F. Everitt, *Ap. J.* **439**, 548 (1995).
3. T. Kaluza, Sitzungsber, Press. Akad. Wiss. Phys. Math. K1, 966 (1921).
4. O. Klein, *Z. Phys.* **37**, 895 (1926).
5. J. Ponce de Leon, *Gen. Rel. Grav.* **20**, 539 (1988).
6. C. Romero, R. Tavakol and R. Zalaletdinov, *Gen. Rel. Grav.* **28**, 365 (1996).
7. P. S. Wesson, *Ap. J.* **394**, 19 (1992).
8. P. S. Wesson and H. Liu, *Ap. J.* **440**, 1 (1995).
9. P. S. Wesson and J. Ponce de Leon, *J. Math. Phys.* **33**, 3883 (1992).
10. P. S. Wesson and J. Ponce de Leon, *J. Astron. Astro.* **294**, 1 (1995).