Induced Matter and Particle Motion in Non-Compact Kaluza-Klein Gravity

A. P. Billyard*
Department of Physics,
Queen's University,
Kingston, Ontario,
Canada K7L 3N6

W. N. Sajko†
Department of Physics,
University of Waterloo,
Waterloo, Ontario,
Canada N2L 3G1

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Abstract

We examine generalizations of the five–dimensional canonical metric by including a dependence of the extra coordinate in the four–dimensional metric. We discuss a more appropriate way to interpret the four–dimensional energy–momentum tensor induced from the five–dimensional space-time and show it can lead to quite different physical situations depending on the interpretation chosen. Furthermore, we show that the assumption of five–dimensional null trajectories in Kaluza–Klein gravity can correspond to either four–dimensional massive or null trajectories when the path parameterization is chosen properly. Retaining the extra–coordinate dependence in the metric, we show the possibility of a cosmological variation in the rest masses of particles and a consequent departure from four–dimensional geodesic motion by a geometric force. In the examples given, we show that at late times it is possible for particles traveling along 5D null geodesics to be in a frame consistent with the induced matter scenario.

Keywords: Kaluza-Klein, induced matter, cosmological constant, geodesics.

*Email: jaf@astro.queensu.ca
†Email: william.sajko@ca.jdsuniphase.com

1 Introduction

The modern version of non-compactified five–dimensional (5D) Kaluza-Klein gravity, in which the 5D cylinder condition ($\partial_4 \hat{g}_{AB} = 0$)[1] has been eliminated in favour of retaining the metric's dependence on the extra coordinate, has had great success in describing four–dimensional (4D) general relativity with an induced energy-momentum tensor (see [1] for a recent review). The 5D space-time can be viewed as a foliation of 4D sheets on which general relativity holds and a stress-energy tensor is induced through the metric dependence on the extra coordinate [2]. This procedure is always mathematically possible due to local embedding theorems which state that a 4D Riemannian manifold (GR) can be locally embedded in a 5D Ricci-flat Riemannian manifold [3, 4].

In the induced–matter scenario, the induced Einstein tensor is typically constructed from 4D metric $g_{\alpha\beta}$ defined by

$$d\hat{s}^2 = g_{\alpha\beta}(x^\Sigma, \ell)dx^\alpha dx^\beta + \epsilon\phi(x^\Sigma, \ell)d\ell^2. \tag{1.1}$$

where the signature of the 4D metric $g_{\alpha\beta}$ is $(+,-,-,-)$; also $x^\Sigma \equiv \{x^\alpha\}$, and $\epsilon \equiv \pm 1$, which leaves the signature of the fifth dimension general and may allow a "two-time" metric (these types of metrics may appear odd but can be shown to give sensible results in the induced-matter context [5, 6]). However, it has been shown [7] that metrics of the "canonical form"

$$d\hat{s}^2 = \frac{l^2}{L^2} g_{\alpha\beta}(x^\Sigma)dx^\alpha dx^\beta - d\ell^2$$

lead to an induced false vacuum equation of state and hence this form naturally leads to an

[1]Throughout this paper we use accent *circumflex* to designate 5D quantities and no accents for 4D quantities; also, uppercase Latin letters are used for the 5D manifold, and lowercase Greek indices are used for the 4D manifold. This paper uses units $8\pi G = c = 1$ unless explicitly stated.

induced cosmological constant, which is parameterized by L. Hence, it would seem that for manifolds of the form

$$d\hat{s}^2 = \frac{l^2}{L^2} g_{\alpha\beta}(x^\Sigma, \ell) dx^\alpha dx^\beta + \epsilon\phi(x^\Sigma, \ell)^2 d\ell^2, \tag{1.2}$$

part of the induced Einstein tensor would have a contribution from an induced cosmological constant, an induced stress-energy from the $\partial_\ell g_{\alpha\beta}$ contributions as well as contributions from the scalar field ϕ.

Closely related to the induced-matter paradigm is the question of the interpretation of 5D geodesics. It has previously been shown [8] that if particles were to follow 5D geodesics, then they cannot in general remain on $\ell = \ell_0$ hypersurfaces. Therefore, the induced stress-energy tensor defined by $g_{\alpha\beta}$ would not be what is observed by an observer moving along 5D geodesics. Within the Space-Time-Matter (STM) theory [9, 10] to give a physically meaningful interpretation to the extra coordinate, ℓ may be interpreted as the rest mass of particles [11] and so the change in the rest–mass of a particle is dictated by the change in ℓ. Because the induced matter is derived from a simple 5D theory, it is tempting to assume that the motion of particles is also naturally 5D (in fact, 5D geodesic since the 5D manifold is a vacuum). However, in general this is incompatible with the induce–matter scenario.

In what follows we first derive the 4D induced energy–momentum tensor from $g_{\alpha\beta}$ in (1.2), decomposing it into a false vacuum component, matter component and scalar field component (if present). In the literature, the induced matter is typically interpreted as either a perfect fluid or a fluid with anisotropic pressures, and we show that these are not the only possible types of matter to model. To demonstrate this we present two examples. We then explore the 5D null geodesic equation and show that these special geodesics can reduce to 4D geodesics for massless particles, but there is an acceleration of massive particles due to a geometric force (which has been previously labeled as a "fifth force" [12]) which depends on a scalar field and has an explicit dependence on the extra dimension. We then elucidate these ideas with same two models and then make our final remarks.

2 4D Induced Matter From 5D Vacuum

We wish to derive the induced matter resulting from the reduction of a 5D vacuum to a 4D hypersurface. Consider the following gauge choice for the 5D metric which explicitly depends on the extra coordinate $x^4 \equiv \ell$, and for which $\hat{g}_{\alpha 4} \propto A_\alpha$ (the electromagnetic vector potential) is set to zero. We factor out a conformal dependence on the 4D metric and include a scalar field so that the 5D metric can be written as

$$\hat{g}_{AB} = \begin{pmatrix} \frac{\ell^2}{L^2}\, g_{\alpha\beta}(x^\Sigma, \ell) & 0 \\ 0 & \epsilon\phi^2(x^\Sigma, \ell) \end{pmatrix} . \tag{2.1}$$

The easiest way to determine the induced matter on the 4D hypersurfaces ($\ell = \ell_o = const.$) is to decompose the 5D metric using a 4+1 decomposition; the "4" is used to designate 4D hypersurfaces with an induced metric $(\ell_o^2/L^2)g_{\alpha\beta}$, and the "1" corresponds to the lapse in the extra dimension between adjacent 4D hypersurfaces measured by the scalar field ϕ. This procedure was initially performed in [2], and for the metric (1.1) the components of the 5D vacuum field equations $\hat{R}_{AB} = 0$ are:

$$\hat{R}_{\alpha\beta} = 0 \quad \Rightarrow \quad R_{\alpha\beta} = \frac{1}{\phi}\nabla_\alpha\nabla_\beta\phi - \frac{\epsilon}{\phi}\partial_\ell K_{\alpha\beta} + \epsilon\left(KK_{\alpha\beta} - 2K_{\alpha\gamma}K^\gamma{}_\beta\right) , \tag{2.2a}$$

$$\hat{R}_{\ell\beta} = 0 \quad \Rightarrow \quad \nabla_\alpha\left(K^\alpha{}_\beta - \delta^\alpha{}_\beta K\right) = 0 \, , \tag{2.2b}$$

$$\hat{R}_{\ell\ell} = 0 \quad \Rightarrow \quad \epsilon\,\Box\phi = \partial_\ell K - \phi K^{\alpha\beta}K_{\alpha\beta} \, , \tag{2.2c}$$

where the covariant derivative and the d'Alembertian operator ($\Box$) are defined on the 4D hypersurfaces. Here the extrinsic curvature of the embedded 4D hypersurfaces is defined as

$$K_{\alpha\beta} \equiv -\frac{1}{2\phi}\partial_\ell \left(\frac{\ell^2}{L^2} g_{\alpha\beta}(x^\Sigma, \ell)\right) , \tag{2.3}$$

and $K \equiv K^\alpha_\alpha = \frac{L^2}{\ell^2} g^{\alpha\beta} K_{\alpha\beta}$. It is evident that the extra coordinate dependence in the 4D metric plays a crucial rôle in inducing matter in 4D. However, if $\partial_\ell\, g_{\alpha\beta} = 0$ then the only consistent solution to the above equations is

$$\partial_\ell\, g_{\alpha\beta} = 0 \quad \Rightarrow \quad R_{\alpha\beta} = \frac{3\epsilon}{L^2}\, g_{\alpha\beta}\,, \quad \phi = 1. \tag{2.4}$$

This solution can be identified as a false vacuum (i.e., $\mu = -p = \Lambda$), provided the constant L is identified with Λ via

$$\Lambda \equiv -\frac{3\epsilon}{L^2}\,. \tag{2.5}$$

The induced cosmological constant generates either the de Sitter vacuum when $\epsilon = -1$ ($\Lambda > 0$) or the anti-de Sitter vacuum when $\epsilon = +1$ ($\Lambda < 0$, which leads to a two-time metric). When the 4D metric depends on ℓ the extra terms generated by the derivatives with respect to the extra coordinate (and possibly the scalar field terms) can be viewed as the matter contribution to the stress-energy, whereas terms proportional to $g_{\alpha\beta}$ can be related to the vacuum stress-energy.

Let us now investigate the matter induced from the energy–momentum tensor derived from $g_{\alpha\beta}(x^\Sigma, \ell)$, assuming that $\phi = \phi(x^\Sigma)$. First, we isolate terms in (2.2a) proportional to $g_{\alpha\beta}$ and identify these terms with the induced effective cosmological "constant", Λ_{eff}. Therefore, we begin with

$$K_{\alpha\beta} = -\frac{\ell}{\phi L^2} g_{\alpha\beta} - \frac{\ell^2}{L^2} J_{\alpha\beta}, \tag{2.6}$$

where

$$J_{\alpha\beta} \equiv \frac{1}{2\phi}\partial_\ell g_{\alpha\beta}. \tag{2.7}$$

Substituting (2.6) into (2.2a) leads to

$$R_{\alpha\beta} = \frac{\nabla_\alpha\nabla_\beta\phi}{\phi} + \frac{3\epsilon}{\phi^2 L^2}\left(1+\frac{1}{3}\phi\ell J\right) g_{\alpha\beta} + \frac{\epsilon\ell^2}{\phi L^2}\left(\frac{4J_{\alpha\beta}}{\ell} + \partial_\ell J_{\alpha\beta} + \phi(JJ_{\alpha\beta} - 2J_{\alpha\gamma}J^\gamma_\beta)\right), \tag{2.8}$$

(where $J \equiv J^\alpha_\alpha = g^{\alpha\beta}J_{\alpha\beta}$) and hence, the effective cosmological constant is defined as

$$\Lambda_{\mathit{eff}} = -\frac{3\,\epsilon}{\phi^2 L^2}\left(1+\frac{\ell}{6}g^{\mu\nu}\partial_\ell g_{\mu\nu}\right). \tag{2.9}$$

The induced Einstein field equations can thus be written

$$G_{\alpha\beta} = {}^{(\phi)}T_{\alpha\beta} + \Lambda_{\mathit{eff}}\; g_{\alpha\beta} + \frac{{}^{(M)}T_{\alpha\beta}}{\phi}, \tag{2.10}$$

where

$$\begin{aligned}
{}^{(\phi)}T_{\alpha\beta} &= \frac{\nabla_\alpha\nabla_\beta\phi}{\phi} - \frac{\Box\phi}{\phi}g_{\alpha\beta}, & \text{(2.11a)}\\
{}^{(M)}T_{\alpha\beta} &= \frac{\epsilon\ell^2}{L^2}\left\{\frac{4J_{\alpha\beta}}{\ell} + \partial_\ell J_{\alpha\beta} + \phi\left(JJ_{\alpha\beta} - 2J_{\alpha\gamma}J^\gamma_\beta\right)\right.\\
&\quad \left. -\frac{1}{2}g_{\alpha\beta}\left[\frac{6J}{\ell} + g^{\mu\nu}\partial_\ell J_{\mu\nu} + \partial_\ell J + \phi\left(J^2 - J^{\mu\nu}J_{\mu\nu}\right)\right]\right\}. & \text{(2.11b)}
\end{aligned}$$

Note that five–dimensional vacuum relativity corresponds to a $\omega = 0$ Brans–Dicke theory [8], which is why we have left an explicit factor of ϕ^{-1} in front of the matter term in (2.10). The case $\phi = 1$ reduces to ordinary 4D relativity with matter.

It is necessary to comment on the kinematic quantities of ${}^{(M)}T^\alpha_\beta$. Often in the literature concerning induced matter from Kaluza–Klein theory, it is often assumed that the induced stress–energy tensor represents either perfect fluid model or a fluid model with anisotropic pressures. However, this is not necessarily the case; indeed the induced stress–energy tensor may not be appropriate for a fluid source at all. To represent a fluid source, the tensor ${}^{(M)}T^{\alpha}_{\beta}$ must be of Segré type {1,1,1,1}; that is, in its Jordan form, ${}^{(M)}T^{\alpha}_{\beta}$ will be diagonal, the components of which will be the eigenvalues of the energy–momentum tensor. One eigenvalue will be associated with a time–like eigenvector and the other three will be associated with space–like eigenvalues. If this is satisfied, then ${}^{(M)}T^{\alpha}_{\beta}$ can be modeled as a fluid with a time–like velocity field u^{α}. If the space–like eigenvectors are all equal then, *and only then*, can the stress tensor be modeled as a perfect fluid. The kinematic quantities $\{\mu, p, u^{\alpha}, q^{\alpha}, \pi^{\alpha}_{\beta}\}$[2] can thus be determined from the eigenvalues and eigenvectors, and Appendix A describes how to compute these quantities for two important cases: fluids with heat conduction and isotropic pressures, $q^{\alpha} \neq 0$ & $\pi^{\alpha}_{\beta} = 0$, and fluids without heat conduction $q^{\alpha} = 0$ & $\pi^{\alpha}_{\beta} \neq 0$. We now present two examples.

2.1 Example A: Ponce de Leon metric

The first example is the one-parameter class of solutions found by Ponce de Leon [13]:

$$d\hat{s}^2 = \frac{\ell^2}{L^2}\left[dt^2 - \left(\frac{t}{L}\right)^{2/\alpha}\left(\frac{\ell}{L}\right)^{2\alpha/(1-\alpha)} d\vec{x}\cdot d\vec{x}\right] - \left(\frac{\alpha}{1-\alpha}\right)^2\left(\frac{t}{L}\right)^2 d\ell^2\,, \tag{2.12}$$

where α is a constant. These solutions have been previously used in a cosmological context since they are spatially isotropic and homogeneous, and on the induced 4D hypersurfaces they are the analogues of the $k = 0$ FRW cosmologies. Using (2.9) for the effective cosmological constant and (2.10) for the induced stress-energy tensors we find that

$$\Lambda_{\mathit{eff}} = \frac{3(1-\alpha)}{(\alpha t)^2} \tag{2.13a}$$

[2] Here, μ is the fluids energy density, p its averaged pressure, q^{α} is the fluids heat conduction vector and π^{α}_{β} is the fluids anisotropic pressure tensor.

$$\text{total stress-energy:} \qquad T^{\alpha}_{\beta} \;=\; \begin{bmatrix} \frac{3}{(\alpha t)^2} & 0 \\ 0 & -\frac{(2\alpha-3)}{(\alpha t)^2}\delta^{i}{}_{j} \end{bmatrix} \tag{2.13b}$$

$$\text{scalar stress-energy :} \qquad {}^{(\phi)}T^{\alpha}_{\beta} \;=\; \begin{bmatrix} \frac{-3}{2\alpha t^2} & 0 \\ 0 & -\frac{1}{2\alpha t^2}\delta^{i}{}_{j} \end{bmatrix} \tag{2.13c}$$

$$\text{matter stress-energy :} \qquad {}^{(M)}T^{\alpha}_{\beta} \;=\; \begin{bmatrix} \frac{9}{2(1-\alpha)Lt} & 0 \\ 0 & \frac{3}{2(1-\alpha)Lt}\delta^{i}{}_{j} \end{bmatrix}, \tag{2.13d}$$

where $\delta^{i}{}_{j}$ is the three–dimensional Kronecker–delta function. We note that the effective cosmological constant decreases as t^{-2} which is compatible with string inspired cosmological theories [14] and scalar-tensor gravity [15] (for an extensive bibliography on variable Λ cosmologies see [16]). This is favourable for inflationary models since the cosmological term is large for early times and then decreases to zero for late times. As is evident from the induced energy–momentum tensor (2.13d), all three space–like eigenvalues are equal and so ${}^{(M)}T^{\alpha}_{\beta}$ can aptly represent a perfect fluid with the energy–density (μ) and pressure (p) given by

$$\mu = \frac{9}{2(1-\alpha)Lt}, \quad p = -\frac{3}{2(1-\alpha)Lt} \qquad \Longrightarrow \qquad p = -\frac{1}{3}\mu. \tag{2.14}$$

Here we see the fluid behaves like a barotropic fluid with a linear equation of state parameter $\gamma = 2/3$. Note that it is necessary to impose that $\alpha \leq 1$ ($\mu \geq 0$) in which case the effective cosmological constant is positive. Furthermore, one could demand that the stress-energy tensor arising from the scalar field (2.13c) satisfies the energy conditions (weak, strong and dominant), in which case $\alpha \leq 0$.

If the *entire* energy–momentum tensor is treated as one fluid, we obtain

$$\mu_{\rm tot} = \frac{3}{(\alpha t)^2}, \quad p_{\rm tot} = \frac{(2\alpha-3)}{(\alpha t)^2} \qquad \Longrightarrow \qquad p_{\rm tot} = \left(\frac{2}{3}\alpha - 1\right)\mu_{\rm tot}. \tag{2.15}$$

which is consistent with that found in [17]. In this case, we have a barotropic fluid with a linear equation of state parameter $\gamma = \frac{2}{3}\alpha$. The strong energy condition ($\mu_{\rm tot} + 3p_{\rm tot} \geq 0$) restricts $\alpha \geq 1$ while the dominant energy condition ($\mu_{\rm tot} \geq |p_{\rm tot}|$) restricts $0 \leq \alpha \leq 3$. As discussed in [17], there are three physically relevant choices for α: $\alpha \in (0,1)$ for inflation, $\alpha = 2$ for radiation, and $\alpha = 3/2$ for dust. For the latter two values, the cosmological constant is negative ($\alpha = 0, 1$ are bifurcation values and must be treated separately).

We present this second interpretation (2.15) to demonstrate how different the induced matter can be when we consider the stress–energy tensor as a conglomerate of three separate sources, but feel the first interpretation (2.14) is more appropriate. First of all, such a decomposition is consistent with how the five–dimensional vacuum theory is mathematically equivalent to four–dimensional Brans–Dicke theory (with or without a cosmological constant). Secondly, by considering the scalar field as a separate source, problems such as the discrepancy between gravitational and inertial mass can be resolved (see, for example, [18]).

2.2 Example B: Shell–like Solutions

The next example is a two-parameter class of spherically symmetric solutions [19]:

$$d\hat{s}^2 = \frac{\ell^2}{L^2}\left(A^2 dt^2 - B^2 dr^2 - C^2 r^2 d\Omega^2\right) - d\ell^2, \tag{2.16}$$

where

$$A = \frac{1}{B} + \frac{k_2 L}{\ell}, \tag{2.17a}$$

$$B = \frac{1}{\sqrt{1 - \frac{r^2}{L^2}}}, \tag{2.17b}$$

$$C = 1 + \frac{k_3 L^2}{r\ell}, \tag{2.17c}$$

(note that this form can be expressed in the original form given in [19] by letting $k_2 \to k_2/k_1$ and $t \to k_1 t$). Since these models have $\phi = 1$ they correspond to 4D relativistic models (as opposed to 4D $\omega = 0$ Brans–Dicke models). These solutions have been termed "shell" solutions since at

$$r = r_C = \frac{|k_3| L^2}{\ell} \tag{2.18}$$

(where $C(r_C) = 0$) the density and pressure of the fluid found in [19] diverged (at $r = r_C$ the surface area of the two sphere, $4\pi r^2 C^2$, is zero and so this may be taken as the origin of the system), and at

$$r = r_A = L\sqrt{1 - \frac{k_2^2 L^2}{\ell^2}}, \tag{2.19}$$

(where $A(r_A) = 0$) the pressure diverges. Note that r_A and r_C coincide at

$$r_0 = \frac{|k_3|}{\sqrt{k_2^2 + k_3^2}} L \tag{2.20a}$$

$$\ell_0 = \sqrt{k_2^2 + k_3^2}\; L, \tag{2.20b}$$

and so for $\{r, \ell\} < \{r_0, \ell_0\}$ we have that $r_A < r_C$; since r_C is defined as the centre of the system, r_A is excluded from the manifold for $\ell < \ell_0$.

The four–dimensional component of this metric is the de Sitter metric when the parameters k_2 and k_3 are both zero; thus because of the dependence on the extra-coordinate, this metric may be interpreted as a generalization to the de Sitter vacuum with an effective cosmological constant. To preserve the signature, the radial coordinate must obey $r < |L|$. Furthermore, we adopt the assumptions used in [19] that $L > 0$, $\ell > 0$, $k_2 < 0$ and $k_3 < 0$.

Now, equations (2.9)-(2.11) reduce to:

$$G^\alpha_\beta = \mathrm{diag}\left[\frac{(1+2C)}{L^2C^2}, \frac{(AB+2C)}{L^2ABC^2}, \frac{C+AB+1}{L^2ABC}, \frac{C+AB+1}{L^2ABC}\right], \tag{2.21a}$$

$$\Lambda_{\mathit{eff}} = \frac{C+2AB}{L^2ABC}, \tag{2.21b}$$

$$^{(M)}T^{\alpha}_{\beta} = \mathrm{diag}\left[\frac{(AB-C^2)}{L^2ABC^2}, \frac{(AB-C^2)+2C(1-AB)}{L^2ABC^2}, \frac{(1-AB)}{L^2ABC}, \frac{(1-AB)}{L^2ABC}\right]. \tag{2.21c}$$

Clearly, the eigenvalues of the induced energy–momentum tensor (2.21c) are (see appendix)

$$\lambda_+ = \frac{(AB-C^2)}{L^2ABC^2}, \tag{2.22a}$$

$$\lambda_- = \frac{(AB-C^2)}{L^2ABC^2} + 2\frac{(1-AB)}{L^2ABC}, \tag{2.22b}$$

$$\lambda_2 = \lambda_3 = \frac{(1-AB)}{L^2ABC}. \tag{2.22c}$$

At this point, one can model $^{(M)}T^{\alpha}_{\beta}$ as an imperfect fluid, but there is no unique choice. However, because $\lambda_2 = \lambda_3$ there are two obvious models from which to choose: a fluid with heat conduction and isotropic pressures ($q^\alpha \neq 0$, ${\pi^\alpha}_\beta = 0$) and a fluid with no heat conduction and anisotropic pressures ($q^\alpha = 0$, ${\pi^\alpha}_\beta \neq 0$).

2.2.1 Heat Conduction with Isotropic Pressure

For the case $q^\alpha \neq 0$ and ${\pi^\alpha}_\beta = 0$, equations (A.15) in the appendix lead to the following kinematic quantities:

$$\mu = \left(\frac{1-AB}{L^2ABC} + 2\frac{AB-C^2}{L^2ABC^2}\right), \tag{2.23a}$$

$$p = -\left(\frac{1-AB}{L^2ABC}\right), \tag{2.23b}$$

$$u^\alpha = \frac{1}{\sqrt{2(1-AB)C}}\left[-\frac{\sqrt{(C+1)(C-AB)}}{A}, \frac{\sqrt{(C-1)(C+AB)}}{B}, 0, 0\right], \tag{2.23c}$$

$$q^\alpha = \frac{\sqrt{(C^2-1)(C^2-A^2B^2)}}{L^2AB\sqrt{2(1-AB)C^5}}\left[-\frac{\sqrt{(C-1)(C+AB)}}{A}, \frac{\sqrt{(C+1)(C-AB)}}{B}, 0, 0\right], \tag{2.23d}$$

$$q^2 = \frac{(1-C^2)(A^2B^2-C^2)}{(L^2ABC^2)^2} \tag{2.23e}$$

(where it can be verified that $u^\alpha u_\alpha = 1$). As discussed in the appendix, ${}^{(M)}T^\alpha_\beta$ will be of Segré type {1,1,1,1} if $(\mu+p)^2 - 4q^2 > 0$ and indeed

$$(\mu+p)^2 - 4q^2 = \frac{4(AB-1)^2}{(L^2ABC)^2} > 0. \tag{2.24}$$

2.2.2 Anisotropic Pressure with No Heat Conduction

For the case where $q^\alpha = 0$ and $\pi^\alpha_\beta \neq 0$, equations (A.16) yield

$$\mu = \frac{(AB-C^2)}{L^2ABC^2}, \tag{2.25a}$$

$$p = -\frac{1}{3}\frac{AB-C^2}{L^2ABC^2} - \frac{4}{3}\frac{(1-AB)}{L^2ABC}, \tag{2.25b}$$

$$u^\alpha = \left[\frac{1}{A}, 0, 0, 0\right] \tag{2.25c}$$

$$\pi^\alpha_\beta = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{2}{3}\frac{(C-1)(C+AB)}{L^2ABC^2} & 0 & 0 \\ 0 & 0 & \frac{1}{3}\frac{(C-1)(C+AB)}{L^2ABC^2} & 0 \\ 0 & 0 & 0 & \frac{1}{3}\frac{(C-1)(C+AB)}{L^2ABC^2} \end{bmatrix}. \tag{2.25d}$$

Clearly, as sections 2.2.1 and 2.2.2 demonstrate, the same 5D metric can yield two very different physical models. In the first case the induced matter is that of a fluid which has heat conducting in the radial direction, whereas the second case is an induced matter without heat conduction but with anisotropic pressures (the radial pressure is different from the solid angle pressure). However, these are not the only possible models from which to choose. For instance, in [20] it has been shown that stress tensors of Segré type $\{1, 1, (1, 1)\}$ can also be used to model a perfect fluid and a electromagnetic field (either null or non–null).

3 Motion, Mass Variation, and the Geometric Force

In this section we approach particle dynamics from a 5D Lagrangian for the canonical metric (2.1) and use the Euler-Lagrange equations to obtain the acceleration equation induced in 4D. When the path parameterization is chosen judiciously we show that the components of the 5D acceleration equation reproduce the 4D geodesic equation for null particles and an acceleration equation for massive particles. With the interpretation is that the extra coordinate is related to the rest-masses of particles [11] the 5D null geodesics lead to a rest-mass variation for massive particles. We elucidate these results with the models studied in 2.1 and 2.2.

3.1 Motion and Mass Variation

To study dynamics in 5D Kaluza-Klein gravity with the canonical metric (2.1) we begin by extremizing the action

$$\hat{I} = \int_A^B \hat{\mathcal{L}}(x^A, \dot{x}^A)\, d\lambda = \int_A^B d\lambda\, \sqrt{\frac{\ell^2}{L^2} g_{\alpha\beta}(x^\Sigma,\, \ell) \frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} + \phi^2(x^\Sigma,\, \ell)\frac{d\ell^2}{d\lambda^2}}, \tag{3.1}$$

where λ is an arbitrary path parameter and the velocities are coterminal at the points A, B. With these boundary conditions, extremizing the action gives the well-known Euler-Lagrange equations

$$\frac{d}{d\lambda}\left(\frac{\partial \hat{\mathcal{L}}}{\partial \hat{u}^A}\right) - \frac{\partial \hat{\mathcal{L}}}{\partial x^A} = 0 \quad \Rightarrow \quad \frac{d\hat{u}^A}{d\lambda} + \hat{\Gamma}^A_{BC}\, \hat{u}^B\, \hat{u}^C = \hat{u}^A \frac{d}{d\lambda}\left(\ln \hat{\mathcal{L}}\right). \tag{3.2}$$

The 4D and ℓ components of equation (3.2) are

$$u^\beta \nabla_\beta u^\alpha = \frac{d}{d\lambda} \ln\left(\frac{\hat{\mathcal{L}}}{\ell^2}\right) u^\alpha - g^{\alpha\beta}\left[\partial_\ell g_{\beta\gamma} u^\gamma + \frac{1}{2}\left(\frac{L\phi}{\ell}\right)^2 \partial_\beta \left(\ln \phi^2\right) \dot{\ell}\right] \dot{\ell} \tag{3.3a}$$

$$\left(\frac{L\phi\dot{\ell}}{\ell}\right)\left\{\Upsilon\left[\ln\left(\frac{L\phi\dot{\ell}}{\ell}\right)\right]^{\cdot}+\frac{\dot{\phi}}{\phi}\left[\Upsilon-\left(\frac{L\phi\dot{\ell}}{\ell}\right)^{2}\right]\right\}$$
$$=\frac{1}{2}\left[\Upsilon-\left(\frac{L\phi\,\dot{\ell}}{\ell}\right)^{2}\right]\left[\frac{2\Upsilon}{\ell}+\partial_{\ell}g_{\alpha\beta}u^{\alpha}u^{\beta}-\left(\frac{L\phi\dot{\ell}}{\ell}\right)^{2}\partial_{\ell}\ln\phi^{2}\right], \tag{3.3b}$$

where a dot is shorthand for $d/d\lambda$. If the parameterization, λ, were chosen to be either the 5D proper distance, $\hat{s}$, or a 5D null parameterization, then term on the right hand side of (3.2) vanishes (and hence (3.2) describe 5D geodesics); however, we have chosen the parameterization to be the 4D proper distance, $\lambda = s$, so that $u_{\alpha}u^{\alpha} \equiv \Upsilon$ (where $\Upsilon = 1$ for timelike paths and $\Upsilon = 0$ for null paths). The extra terms on the right hand side of the equations (3.3) are a consequence of this choice rather than the 5D proper distance $\lambda = \hat{s}$ [12]. Solving equation (3.3b) for $\dot{\ell}$ is very complicated, and in general the quantities $\{g_{\alpha\beta}, \phi\}$ would have to be first specified. However, from (3.3b) it is apparent that the solution

$$\left(\frac{\dot{\ell}}{\ell}\right)^{2}=\frac{\Upsilon}{L^{2}\phi^{2}} \tag{3.4}$$

satisfies (3.3b) identically for any $\{g_{\alpha\beta}, \phi\}$. It may be verified that (3.4) represents 5D null geodesics by examining the 5D canonical line element (1.2). Hence, the particle paths are consequently 5D null even though we have chosen the 4D proper distance $\lambda = s$ to be the path parameter.

Relation (3.4) constrains the velocity $\dot{\ell}$ but does not give it physical meaning; for this, we turn to Kaluza-Klein theories in which the extra coordinate can be interpreted as a geometric mass via $\ell = Gm/c^2$ [1, 9–11, 21]. We now look at the variation of rest mass as a function of the 4D path parameterization. The rest mass of a particle is easily obtained from integrating

(3.4):

$$m = m_o \exp\left(\pm\sqrt{\frac{\Upsilon}{L^2}}\int ds\,\phi^{-1}\right). \tag{3.5}$$

Since in 4D we have $\Upsilon = 0$ for photons, this implies that the variation in a photon's rest mass is zero and so its mass may consistently be set to zero. However, for 4D paths which have $\Upsilon = 1$, there is a variation in the rest-mass of massive particles driven by the scalar field ϕ and hence ϕ may be modeled as a Higgs-type field. Let us make a few comments:

1. A conformal transformation of the 4D metric $g \to \tilde{g} = \phi^2 g$ would remove the scalar-field dependence in (3.5), but also changes the induced-matter field equations as well as the 4D acceleration $a^\alpha = u^\mu \nabla_\mu u^\alpha$, and complicates matters substantially.

2. When the 4D condition $\partial_\ell\, g_{\alpha\beta} = 0 \to \phi = 1$ is imposed, we get a cosmological variation of the rest masses of massive particles in the de Sitter vacuum ($\epsilon = -1$), namely that

$$m = m_0 e^{\pm(s-s_0)/L}. \tag{3.6}$$

3. If we choose a two-time metric ($\epsilon = +1$) the variation is imaginary, giving an oscillating rest mass in the anti-de Sitter vacuum (this oscillation will hold even for more complicated metrics which do not obey $\partial_\ell\, g_{\alpha\beta} = 0$).

We now turn our attention to the acceleration equation (3.3a). After some algebra, equation (3.3a) reduces to the form

$$u^\beta \nabla_\beta\, u^\alpha = f^\alpha\,, \tag{3.7}$$

where f^α is the force per unit rest mass

$$f^\alpha = -h^{\alpha\gamma}\left(\Upsilon\frac{\phi_\gamma}{\phi} + \partial_\ell\, g_{\gamma\beta}\, u^\beta\, \dot{\ell}\right)\,, \tag{3.8}$$

and $h^{\alpha\gamma} \equiv g^{\alpha\gamma} - u^\alpha u^\gamma$ is the projection tensor. When $\partial_\ell \, g_{\alpha\beta} = 0$ ($\phi = 1$), this force term vanishes, and the motion is geodesic for both photons and massive particles in a pure 4D de Sitter vacuum, which is the correct 4D result in general relativity. However, when $\partial_\ell \, g_{\alpha\beta} \neq 0$, photons will still travel along null 4D geodesics since they obey $\Upsilon = 0$ and $\dot{\ell} = 0$; but massive particles will experience a geometric force since $\Upsilon = 1$ and $\dot{\ell} \neq 0$. We now consider some examples to elucidate these ideas.

3.2 Example A: Ponce de Leon solutions

In this section we revisit the example first discussed in section 2.1. We will show that the rest masses of particles may vary in a cosmological frame which employs a comoving coordinate system, and make some comments about the observability of the geometric force.

Since the 4D metric of (2.12) has a non-trivial ℓ-dependence, Λ_{eff} is not constant. Furthermore, there is a non-trivial mass-variation, and equation (3.4) reduces to the following rest-mass variation (by identifying ℓ with m):

$$\frac{\dot{m}}{m} = \pm \left(\frac{1-\alpha}{\alpha} \right) \frac{1}{t}. \tag{3.9}$$

Assuming $t \sim 10^9 \, yr$ as an order of magnitude for the age of the Universe [22, 23], we find that for $\alpha \lesssim 1$ the variation of rest masses is less than $10^{-11} \, yr^{-1}$ which is consistent with the classical tests of 4D general relativity [1, 24].

The acceleration equation for the Ponce de Leon metric is simplified by the comoving coordinate system. In general, the assumption that the spatial velocities are constant ($u^i = 0$), implies that the scalar field can only depend on time, so $\phi = \phi(t)$. Thus we can conclude that any 5D metric in the canonical form of (2.1), which has the 4D section $g_{\alpha\beta}(x^\Sigma, \ell)$ written in comoving coordinates with a time-dependent scalar field, will not impart a geometric force and the motion will be 4D geodesic. This applies to any spatially isotropic and homogeneous model (i.e., most cosmological models) wherein comoving coordinates may be employed.

3.3 Example B: Shell-like Solutions

For the class of solutions (2.16), $\phi = 1$ and thus

$$\frac{\dot{\ell}}{\ell} = \pm\frac{\sqrt{\Upsilon}}{L} \quad \Leftrightarrow \quad \dot{\ell} \propto e^{\pm\sqrt{\Upsilon}(s-s_0)/L}. \tag{3.10}$$

Hence, this allows $\dot{\ell} \to 0$ at late times and so the geometric force acting on the 4D particle motion can exponentially decay in proper time, s; particles following this motion will asymptote toward $\ell = \ell_0$.

In order to explicitly calculate the geometric force (3.8), we first need to determine the four–dimensional velocities u_α. These can be obtained either from solving (3.7) or deriving them from the 5D geodesics. Since the five-dimensional manifold is Riemann flat, the 5D geodesics are easily obtainable, and it can be shown that the 5D null geodesics are satisfied by

$$\begin{aligned}
\frac{dt}{d\hat{s}} &= \frac{L^2 E}{\ell^2 A^2}, && (3.11a)\\
\frac{d\varphi}{d\hat{s}} &= \frac{L^2 \mathcal{J}}{\ell^2 r^2 C^2}, && (3.11b)\\
\frac{dr}{d\hat{s}} &= \frac{1}{lB^2}\left(L\sqrt{Q_1} - \varepsilon r B\sqrt{Q_2}\right), && (3.11c)\\
\frac{d\ell}{d\hat{s}} &= \frac{1}{LB}\left(rB\sqrt{Q_1} + \varepsilon L\sqrt{Q_2}\right), && (3.11d)
\end{aligned}$$

where

$$\begin{aligned}
Q_1 &= \zeta_0^2 - \frac{L^2\mathcal{J}^2}{\ell^2 r^2 C^2}, && (3.12a)\\
Q_2 &= \frac{L^2 E^2}{\ell^2 A^2} - \zeta_0^2, && (3.12b)
\end{aligned}$$

$\varepsilon^2 = 1$, and $\{E, \mathcal{J}, \zeta_0\}$ are integration constants in which E may be interpreted as the energy per unit rest mass and $\mathcal{J}$ is the angular momentum per unit rest mass (note that we have consistently chosen the declination angle to be $\theta = \pi/2$ with $\frac{d\theta}{d\hat{s}} = 0$).

To obtain the 4D velocities, it is easy to show from the line element of this space–time

$$d\hat{s}^2 = \frac{\ell^2}{L^2} ds^2 - d\ell^2$$

that for 5D null geodesics

$$\frac{d\hat{s}}{ds} = \frac{\ell\sqrt{\Upsilon}}{\phi L}\left(\frac{d\ell}{d\hat{s}}\right)^{-1}, \qquad \Longrightarrow \qquad \dot{x}^\alpha = \frac{\ell\sqrt{\Upsilon}}{\phi L}\frac{dx^\alpha}{d\hat{s}}\left(\frac{d\ell}{d\hat{s}}\right)^{-1}, \tag{3.13}$$

and therefore, the 4D velocities for massive particles ($\Upsilon = 1$) are

$$\dot{t} = \frac{L^2 B E}{\ell A^2\left[rB\sqrt{Q_1} + \varepsilon L\sqrt{Q_2}\right]}, \tag{3.14a}$$

$$\dot{\varphi} = \frac{L^2 B \mathcal{J}}{\ell r^2 C^2\left[rB\sqrt{Q_1} + \varepsilon L\sqrt{Q_2}\right]}, \tag{3.14b}$$

$$\dot{r} = \frac{1}{B}\left(\frac{L\sqrt{Q_1} - \varepsilon rB\sqrt{Q_2}}{rB\sqrt{Q_1} + \varepsilon L\sqrt{Q_2}}\right). \tag{3.14c}$$

It is apparent from these velocities that there will in general be a geometric force acting on massive particles,

$$u^\beta \nabla_\beta u^\alpha = -2h^{\alpha\gamma}\left[\delta^0_\gamma \; |k_2| \; A\dot{t} - \delta^3_\gamma \; |k_3| \; \frac{L}{\ell} \; rC\dot{\varphi}\right]\frac{\dot{\ell}}{\ell}. \tag{3.15}$$

As is evident in (3.15), we clearly see a drag force in the ϕ direction, which is odd for spherically-symmetric solutions. However, this is not unique to this particular solution and from equation (3.8) it is apparent that there will in general be such drag terms as long as the angular part of the metric has dependence in the extra coordinate.

4 Final Comments

By retaining the extra coordinate $x^4 = \ell$ in 5D Kaluza-Klein gravity we have seen that a 5D vacuum induces non-trivial matter on 4D hypersurfaces $\ell = \ell_o$, in which we retrieve a component which acts as a cosmological "constant", a component which can be modeled as a fluid and a scalar field contribution (if present). Rather than considering all three components as a single fluid source, we feel it is important to keep the components distinct because of the close connection between 5D vacuum relativity and 4D general relativity with matter and a scalar field (see [8]), and because the discrepancies between gravitation and inertial masses do not arise when one considers the scalar field separately (see [25]). In particular, we use the eigenvalues and eigenvectors of the induced energy–momentum tensor to properly interpret the induced matter. However, the induced stress–energy tensor does not in general uniquely determine the matter content and the interpretation chosen (for example, whether to model it as with heat conduction or not, etc.) can lead to quite different kinematic quantities.

We have shown that the assumption of 5D null geodesics can lead to a variable rest mass for massive particles, once we identify the extra dimension with mass. The existence of a scalar field could be inferred from particle motion in the coming Satellite Test of the Equivalence Principle (STEP) [26], and consequently any such scalar field would place constraints on the rest mass variation. The acceleration for null particles remained the same as in regular 4D relativity, but the motion for massive particles was augmented by an additional force. This force has a contribution from a scalar field and crucially depends on the existence of the extra dimension. This motion was investigated for the Ponce de Leon class of solutions with a particular extra coordinate dependence that induced a time-varying cosmological constant $\Lambda \sim t^{-2}$. For this metric there is no fifth force due to the nature of the comoving coordinate system. Indeed, for any metric which allows comoving coordinates there will be no fifth force and so a majority of simple cosmological models (e.g., non-tilting models, etc.) in general will appear to allow motion which is geodesic in 4D. For the Shell–like solutions, the 4D motion of particles derived from 5D null geodesics indeed asymptote to 4D geodesics since $\dot{\ell} \to 0$ exponentially. In this example, the induced–matter is not what would be observed by particles traveling along 5D null geodesics until at late times. Indeed, the two paradigms (induced matter and 5D geodesics) are distinct. One cannot say in general that particles traveling along 5D null geodesics will observe the induced matter derived from $\ell = \ell_0$ hypersurfaces, but as the Shell–like solutions demonstrate, it may be possible that at some point in the particle's path (early proper times, late proper times, etc) that the two theories will indeed coincide.

Furthermore, any "angular" drag force terms which arise, as demonstrated in the Shell–like solutions, would induce motion which deviates from that of classical 4D motion and thus provides constraint on this theory. For example, the absence of drag terms in the angular direction in 4D motions suggest that an appropriate 5D metric should be independent of the extra coordinate in the angular components. It seems that we should turn to ℓ-dependent analogues of the Schwarzschild metric to observe and test any deviations from the classical tests of GR due to the fifth force. Work on this is under way, and we expect to relate 5D dynamics to the upcoming Space Test of the Equivalence Principle. Finally, it is important to note that the 4D velocities of the test particles derived from the 5D motion *do not* correspond to the velocities of the induced fluids, although this has often been assumed in the past (for a full discussion, see [27]), but rather they should be interpreted as the velocity of a test particle traveling through the fluid. This is a consistent interpretation within regular GR in which geodesics are assumed for test particles traveling through a fluid [28, Ch. 5.3]

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A Extracting Kinematic Quantities from the Energy–Momentum Tensor

This appendix describes how kinematic variables can be obtained from eigenvalues and eigenvectors of the energy-momentum tensor for non-perfect fluids, generalizing the work found in [29, Chapter 5.1]. Various energy-momentum tensors have physical restrictions based on their Segré type [20, 30]. One must insist that the metric's determinant be Lorentzian ($\det g_{\alpha\beta} < 0$), which eliminates Segré types $\{22\}$ and $\{4\}$ [20, 30]. Furthermore, the strong energy condition, $T^{\alpha}_{\beta} t^{\beta} t_{\alpha} > 0$ (where t^{β} is any time–like vector) eliminates Segré types $\{z, \bar{z}, 1, 1\}$ and $\{3, 1\}$. Finally, the *only* Segré type that admits a time–like eigenvector is $\{1, 1, 1, 1\}$ and its degeneracies, which is necessary for considering fluids with a time–like velocity, u^a. Thus, the main focus here will be on energy-momentum tensors of Segré type $\{1,1,1,1\}$ and its degeneracies.

We begin by assuming the standard non-perfect fluid energy momentum tensor,

$$T^{\alpha}_{\beta} = (\mu + p)\, u^{\alpha}u_{\beta} - p\delta^{\alpha}_{\beta} + u^{\alpha}q_{\beta} + u_{\beta}q^{\alpha} + \pi^{\alpha}_{\beta}, \tag{A.1}$$

where u^{α} is the fluids velocity field, μ is the fluid's energy density, p is the averaged pressure, q^{α} is the heat conduction vector and π^{α}_{β} is the anisotropic pressure tensor. These quantities are constrained by

$$u^{\beta}\pi^{\alpha}_{\beta} = 0, \quad u^{\beta}q_{\beta} = 0, \quad \pi^{\alpha}_{\alpha} = 0, \quad u^{\alpha}u_{\alpha} = 1, \quad q^{\alpha}q_{\alpha} \equiv -q^2. \tag{A.2}$$

One could equally write

$$T^{\alpha}_{\beta} = (\mu + p - \zeta\theta)\, u^{\alpha}u_{\beta} - (p - \zeta\theta)\delta^{\alpha}_{\beta} + u^{\alpha}q_{\beta} + u_{\beta}q^{\alpha} - 2\eta\sigma^{\alpha}_{\beta}, \tag{A.3}$$

to introduce the velocity's shear tensor, σ^{α}_{β}, its expansion scalar, θ, as well as the fluid's bulk viscosity coefficient, ζ, and its shear viscosity, η. One must be careful here, since it is the velocity alone which determines σ^{α}_{β} and θ, and so if one may indeed have $\sigma^{\alpha}_{\beta} \neq 0$ even if it was initially assumed to be zero. The expansion term may be "absorbed" by letting $p = \tilde{p} + \zeta\theta$, and so this term can never be determined from the eigenvalues of T^{α}_{β} alone. Therefore, the form (A.1) will be used throughout. Should $\sigma^{\alpha}_{\beta} \neq 0$ and $\pi^{\alpha}_{\beta} \propto \sigma^{\alpha}_{\beta}$, then the shear viscosity coefficient, η can also be calculated.

If one assumes that $\pi^{\alpha}_{\beta} \neq 0$, then it will have three eigenvectors associated with its principle axes: $v^{\alpha}_{(i)}$ ($i = \{1, 2, 3\}$), where $v^{\alpha}_{(i)}v_{(i)\alpha} = -1$, and $v^{\alpha}_{(i)}v_{(j)\alpha} = 0$ for $i \neq j$. Therefore, we write $\pi^{\alpha}_{\beta} = \sum_i \pi_i v^{\alpha}_{(i)}v_{(i)\beta}$ (summation over for i will remain explicit). Since $\pi^{\alpha}_{\alpha} = 0$ then $\pi_1 + \pi_2 + \pi_3 = 0$ and there are only two independent values for $\{\pi_1, \pi_2, \pi_3\}$. Hence, we will assume the the three space–like eigenvectors can be written in terms of these three vectors.

Should there be no anisotropies, $v^a_{(1)}$ can be used to denote the direction of $q^\alpha = q_1 v^\alpha_{(1)}$ and $v^\alpha_{(2)}$, $v^\alpha_{(3)}$ will be the (eigen)vectors perpendicular to q^α and u^α.

All eigenvectors, y^α, will contain p, so to reduce computation we will define

$$\tilde{T}^\alpha_\beta \equiv T^\alpha_\beta + p\delta^\alpha_\beta, \tag{A.4a}$$

$$\tilde{\lambda} \equiv \lambda + p, \tag{A.4b}$$

where $\tilde{\lambda}$ is defined by

$$\tilde{T}^\alpha_\beta y^\beta = \tilde{\lambda} y^\alpha. \tag{A.5}$$

Hence, for $q^\alpha = \sum_i q_i v^\alpha_{(i)}$ and $\pi^\alpha_\beta = \sum_i \pi_i v^\alpha_{(i)} v_{(i)\beta}$, we have:

$$\tilde{T}^\alpha_\beta u^\beta = (\mu + p)u^\alpha + q_1 v^\alpha_{(1)} + q_2 v^\alpha_{(2)} + q_3 v^\alpha_{(3)}, \tag{A.6a}$$

$$\tilde{T}^\alpha_\beta v^\beta_{(i)} = -\left[q_i u^\alpha + \pi_i v^\alpha_{(i)}\right] \tag{A.6b}$$

$$\tilde{T}^\alpha_\beta q^\beta = -\left[q^2 u^\alpha + \sum_{i=1}^{3} q_i \pi_i v^\alpha_{(i)}\right], \tag{A.6c}$$

where $q^2 = q_1^2 + q_2^2 + q_3^2$. If one multiplies each equation of (A.6b) with q_i and sum, one yields equation (A.6c) and so the last may be omitted when considering $q^\alpha \neq 0$, $\pi^\alpha_\beta \neq 0$. However, in the event that $\pi^\alpha_\beta = 0$ or $q^\beta \pi^\alpha_\beta = 0$, then one may take $v^\alpha_{(1)}$ as the direction of q^α and the other two perpendicular to $v^\alpha_{(1)}$, and so the first equation of (A.6b) may be replaced by (A.6c).

In general, we seek eigenvectors of the form

$$\chi^\alpha = a u^\alpha + b v^\alpha_{(1)} + c v^\alpha_{(2)} + d v^\alpha_{(3)}, \tag{A.7}$$

where $\{a, b, c, d\} \in \mathbb{R}$. Hence, (A.5) yields the four equations:

$$a\tilde{\lambda} = a(\mu + p) - bq_1 - cq_2 - dq_3, \tag{A.8a}$$

$$b\tilde{\lambda} = aq_1 - b\pi_1, \tag{A.8b}$$

$$c\tilde{\lambda} = aq_2 - c\pi_2, \tag{A.8c}$$

$$d\tilde{\lambda} = aq_3 - d\pi_3. \tag{A.8d}$$

Here, we have four equations for five unknowns $\{\tilde{\lambda}, a, b, c, d\}$ and so we may arbitrarily set one to a particular value (say, to normalize the vector). This is a reflection of the fact that eigenvectors can be arbitrarily scaled without affecting (A.5). Although this may make the system determined, we need to express the seven quantities $\{\mu, p, q_1, q_2, q_3, \pi_1, \pi_2\}$ in terms of the four eigenvalues, and so we would then need auxiliary equations (at most 3) to specify all parameters. However, we shall only consider here two cases, $\pi^\alpha_\beta = 0$, $q^\alpha \neq 0$ and $\pi^\alpha_\beta \neq 0$ and $q^\alpha = 0$, and for these cases the system is closed.

A.1 Case 1: $\pi^a_b = 0$

For $\pi^\alpha_\beta = 0$, it has been shown that T^α_β has to be of Segré type $\{1, 1, (1, 1)\}$ [30] (providing that $(\mu+p)^2+4q^2 > 0$), with two degenerate eigenvalues. Here, the eigenvectors $v^\alpha_{(2,3)}$ will be orthogonal to u^α and q^α with eigenvalues $\lambda_2 = \lambda_3 = -p$. We then need to find the two other eigenvectors $\chi^\alpha_\pm$ and their corresponding eigenvectors $\tilde{\lambda}_\pm$. In this case, we may let $b \to bq_1$ ($q_1^2 = q^2$) and consider only (A.8a) and (A.8b):

$$\begin{aligned} a\tilde{\lambda}_\pm &= a(\mu + p) - bq^2 \\ b\tilde{\lambda}_\pm &= a. \end{aligned}$$

The solutions to these equations are

$$\frac{a_\pm}{b_\pm} = \tilde{\lambda}_\pm = \frac{1}{2}(\mu + p) \pm \frac{1}{2}\sqrt{(\mu + p)^2 - 4q^2} \tag{A.9a}$$

$$\lambda_\pm = -\frac{1}{2}(p - \mu) \pm \frac{1}{2}\sqrt{(\mu + p)^2 - 4q^2}. \tag{A.9b}$$

Defining,

$$\begin{aligned}
\Delta\lambda &\equiv \lambda_+ - \lambda_- = \sqrt{(\mu+p)^2 - 4q^2}, && \text{(A.10a)}\\
\bar{\lambda} &\equiv \frac{1}{2}(\lambda_+ + \lambda_-) = -\frac{1}{2}(p-\mu), && \text{(A.10b)}\\
\Lambda &\equiv \lambda_2 - \bar{\lambda} = -\frac{1}{2}(p+\mu), && \text{(A.10c)}
\end{aligned}$$

the magnitudes of $\chi_\pm^a$ are

$$\frac{\chi_+^2}{b_+^2} \qquad \begin{cases} = \frac{1}{2}\left[(\mu+p)^2 - 4q^2\right] + \frac{1}{2}(\mu+p)\sqrt{(\mu+p)^2-4q^2} \\ = \frac{1}{2}\Delta\lambda^2 - \Lambda\Delta\lambda \end{cases}, \tag{A.11}$$

$$\frac{\chi_-^2}{b_-^2} \qquad \begin{cases} = \frac{1}{2}\left[(\mu+p)^2 - 4q^2\right] - \frac{1}{2}(\mu+p)\sqrt{(\mu+p)^2-4q^2} \\ = \frac{1}{2}\Delta\lambda^2 + \Lambda\Delta\lambda \end{cases}, \tag{A.12}$$

$$\chi_\pm^\alpha \chi_{\mp\alpha} = 0, \tag{A.13}$$

where it may be shown that $|\Lambda| > \frac{1}{2}|\Delta\lambda|$ for $q^2 > 0$, and so χ_+^α is time–like ($\chi_+^2 > 0$) and χ_-^α is space–like ($\chi_-^2 < 0$). Hence, by defining

$$b_\pm^{-2} = -\Lambda\Delta\lambda \pm \frac{1}{2}\Delta\lambda^2 > 0, \tag{A.14}$$

we normalize these vectors to $\chi_\pm^\alpha \chi_{\pm\alpha} = \pm 1$.

The kinematic variables are, in terms of the eigenvalues/eigenvectors,

$$\begin{aligned}
\mu &= \lambda_+ + \lambda_- - \lambda_2, && \text{(A.15a)}\\
p &= -\lambda_2, && \text{(A.15b)}\\
q^2 &= \left[\lambda_2 - \lambda_+\right]\left[\lambda_2 - \lambda_-\right], && \text{(A.15c)}
\end{aligned}$$

$$
\begin{aligned}
u^\alpha &= \frac{1}{\Delta\lambda}\left(\frac{\chi_+^\alpha}{b_+} - \frac{\chi_-^\alpha}{b_-}\right), && \text{(A.15d)}\\
q^\alpha &= \frac{1}{2}\left(\frac{\chi_-^\alpha}{b_-} + \frac{\chi_+^\alpha}{b_+}\right) + \frac{\Lambda}{\Delta\lambda}\left(\frac{\chi_+^\alpha}{b_+} - \frac{\chi_-^\alpha}{b_-}\right). && \text{(A.15e)}
\end{aligned}
$$

As evident from the magnitudes of $\chi_\pm^a$ we have the following cases:

1. $(\mu+p)^2 > 4q^2$: χ_-^α is time–like and χ_+^α is space–like. Segré type $\{1, 1, (1, 1)\}$; physically relevant.

2. $(\mu + p)^2 = 4q^2$: $\lambda_+ = \lambda_-$ and $\chi_\pm^\alpha$ are null. Segré type $\{2, (1, 1)\}$.

3. $(\mu + p)^2 < 4q^2$: $\lambda_\pm$ and $\chi_\pm^\alpha$ are complex. Segré type $\{z, \bar{z}, (1, 1)\}$.

A.2 Case 2: $q^\alpha = 0$

This case is fairly simple, since $v_{(i)}^\alpha u_\alpha = 0$ and $v_{(i)}^\alpha v_{(j)\alpha} = 0$, and so it is quite apparent that $\{u^\alpha, v_{(1)}^\alpha, v_{(2)}^\alpha, v_{(3)}^\alpha\}$ are eigenvectors (see equations (A.6a) and (A.6b)). Denoting λ_0 to be the eigenvalue associated with u^α, one has

$$
\begin{aligned}
\mu &= \lambda_0 && \text{(A.16a)}\\
p &= -\frac{1}{3}\left(\lambda_1 + \lambda_2 + \lambda_3\right) && \text{(A.16b)}\\
\pi_i &= -\lambda_i - p = -\left[\lambda_i - \frac{1}{3}\sum_{j=1}^{3}\lambda_j\right] && \text{(A.16c)}
\end{aligned}
$$

If one finds $\sigma_\beta^\alpha \neq 0$ and $\sigma_\beta^\alpha \propto \pi_\beta^\alpha$ then the shear viscosity coefficient may be determined via

$$
\eta = -\frac{\pi_1}{2\sigma_1} = -\frac{\pi_2}{2\sigma_2} = -\frac{\pi_3}{2\sigma_3}, \tag{A.17}
$$

where σ_i are the eigenvalues of the shear tensor.

References

[1] Overduin, J. and Wesson, P. S. (1997). *Phys. Rep.* 283, 303.

[2] Sajko, W. N., Wesson, P. S., and Liu, H. (1998). *J. Math. Phys* 39, 2193.

[3] Rippl, S., Romero, C., and Tavakol, R. (1995). *Class. Quantum Grav.* 12, 2411.

[4] Romero, C., Tavakol, R., and Zalaletdinov, R. (1996). *Gen. Rel. Grav.* 28, 365.

[5] Billyard, A. P. and Wesson, P. S. (1996). *Gen. Rel. Grav.* 28, 129.

[6] Billyard, A. P. and Wesson, P. S. (1996). *Phys. Rev. D* 53, 731.

[7] Mashhoon, B., Liu, H., and Wesson, P. S. (1994). *Phys. Lett. B* 331, 305.

[8] Billyard, A. P. and Coley, A. A. (1997). *Mod. Phys. Lett. A* 12, 2121.

[9] Wesson, P. S., Ponce de Leon, Liu, H., Masshoon, B., Kalligas, D., Everitt, C. W. F., Billyard, A., Lim, P., and Overduin, J. (1996). *Int. J. Mod. Phys. A* 11, 3247.

[10] Wesson, P. S. (1999). *Space, Time, Matter: Modern Kaluza-Klein Theory*. (World Scientific, River Edge, New Jersey).

[11] Wesson, P. S. (1984). *Gen. Rel. Grav.* 16, 193.

[12] Mashhoon, B., Wesson, P. S., and Liu, H. (1998). *Gen. Rel. Grav.* 30, 555.

[13] Ponce de Leon, J. (1988). *Gen. Rel. Grav.* 20, 539.

[14] Lopez, J. L. and Nanopoulos, D. V. (1996). *Mod. Phys. Lett. A* 11, 1.

[15] Endo, M. and Fukui, T. (1977). *Gen. Rel. Grav.* 8, 833.

[16] Overduin, J. and Cooperstock, F. I. (1998). *Phys. Rev. D* 58, 043506.

[17] Wesson, P. S. (1992). *Astrophys. J.* 394, 19.

[18] Sajko, W. N. (1999). *Phys. Rev. D* 60, 104038.

[19] Wesson, P. S. and Liu, H. (1998). *Phys. Lett. B* 432, 266.

[20] Hall, G. S. (1984). *Arab. J. Sci. Eng.* 9, 87.

[21] Wesson, P. S., Mashhoon, B., and Liu, H. (1997). *Mod. Phys. Lett. A* 12, 2309.

[22] Chaboyer, B. (1998). *Phys. Rept.* 307, 23.

[23] Chaboyer, B., Demarque, P., Kernan, P. J., and Krauss, L. M. (1998). *Astrophys. J.* 494, 96.

[24] Bekenstein, J. D. (1977). *Phys. Rev. D* 15, 1458.

[25] Sajko, W. N. (2000). *Int. J. Mod. Phys. D* 9, 445.

[26] Reinhard, R., Jafry, Y., and Laurance, R. (1993). *Euro. Space Agency Jour.* 17, 251.

[27] Billyard, A. P. and Coley, A. A. (1997). *Mod. Phys. Lett. A* 12, 2223.

[28] Wald, R. M. (1984). *General Relativity*. (The University of Chicago Press, Chicago, Illinois).

[29] Kramer, D., Stephani, H., Herlt, E., and MacCallum, M. A. H. (1980). *Exact Solutions of Einstein's Field Equations*. (Cambridge University Press, Cambridge).

[30] Hall, G. S. (1993). *Symmetries in General Relativity*. Monograph series, Centro Brasileiro de Pesquisas Fı́icas. CBPF-MO-001/93.