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Inhomogeneous cosmological models in scalar-tensor theories

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Abstract. – Exact self-similar spatially inhomogeneous G_2 cosmological models are found in a class of scalar-tensor theories of gravity by exploiting the formal equivalence of this class of theories (under a conformal transformation and field redefinition) to general relativity minimally coupled to a scalar field with an exponential potential. We argue that these exact self-similar solutions may play an important role in describing the asymptotic behaviour (both at early and at late times) of more general scalar-tensor G_2 models. The possible isotropization and the homogenization of these models is briefy discussed.

In a companion paper [1], we studied the class of scalar-tensor theories of gravity with the action [2, 3]

$$\overline{S} = \int \sqrt{-\bar{g}} \left[\phi \overline{R} - \frac{\omega(\phi)}{\phi} \bar{g}^{ab} \phi_{,a} \phi_{,b} - 2\overline{V}(\phi) \right] \mathrm{d}^4 x \,, \tag{1}$$

which is equivalent, under the conformal transformation and field redefinition [4-7]

$$g_{ab} = \phi \bar{g}_{ab} \,, \tag{2a}$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\phi} = \frac{\pm\sqrt{\omega(\phi) + 3/2}}{\phi} \,, \tag{2b}$$

to general relativity coupled to a scalar field with the action

$$S = \int \sqrt{-g} \left[R - g^{ab} \varphi_{,a} \varphi_{,b} - 2\frac{\overline{V}(\phi)}{\phi^2} \right] \mathrm{d}^4 x \,. \tag{3}$$

In particular, we studied those theories that transform into (3) such that the potential of the scalar field is of the exponential type, *viz*

$$V(\varphi) = V_0 e^{k\varphi} = \frac{\overline{V}(\phi(\varphi))}{\phi^2(\varphi)} \,, \tag{4}$$

where V_0 and k are (non-negative) constants, and we exploited previous results on the asymptotic behaviour of spatially homogeneous scalar-field cosmological models with an exponential potential [8-10] to study the asymptotic properties of the corresponding models in scalar-tensor theories. In this brief report, we shall extend this study to a class of inhomogeneous scalar-tensor models using similar techniques. In the last years there has been an interest in studying inhomogeneous cosmological solutions of low-energy string theory [11]. Since the low-energy action for bosonic string theory is identical to the JBD action with $\omega = -1$, it seems worth extending the analysis done in [1] to inhomogeneous scalar-tensor models. As a first approach to this problem we will try to benefit from previous results obtained by studying the asymptotic behaviour of a certain class of inhomogeneous metrics [12].

Indeed, we shall consider the subclass of scalar-tensor theories of gravity in which the arbitrary functions are given by

$$\omega(\phi) = \omega_0 \,, \tag{5a}$$

$$\overline{V} = \beta\phi^{\alpha}, \tag{5b}$$

where ω_0, α and β are constants (*i.e.*, the potential V is of power law form), so that eq. (2b) integrates to yield

$$\phi = \phi_0 \ \exp\left[\frac{\varphi - \varphi_0}{\bar{\omega}}\right] , \tag{6}$$

where

$$\bar{\omega} \equiv \pm\sqrt{\omega_0 + 3/2}, \tag{7}$$

and hence

$$V = V_0 e^{k\varphi} \,, \tag{8}$$

where

$$k = \frac{\alpha - 2}{\bar{\omega}} \,. \tag{9}$$

Clearly, the corresponding action (3) represents general relativity coupled to a scalar field with an exponential potential.

We shall study the scalar-tensor theories (1) and (5) in the inhomogenous G_2 geometry in which there exist two commuting space-like Killing vectors ($\partial/\partial\mathbf{x}$ and $\partial/\partial\mathbf{y}$) and the metric is given by

$$\mathrm{d}s^2 = e^F\left(-\mathrm{d}t^2 + \mathrm{d}z^2\right) + G\left(e^p\mathrm{d}x^2 + e^{-p}\mathrm{d}y^2\right) \,, \tag{10}$$

where all metric functions depend upon t and z [13]. To preserve the G_2 geometry the scalar field is assumed to be of the form

$$\phi = \phi(t, z) \,, \tag{11}$$

so that providing the transformation (2a) is non-singular, the corresponding general relativistic metric $\boldsymbol{g}$ is also a G_2 metric. G_2 scalar-field cosmological models with an exponential potential have been studied by Ibáñez and Olasagasti [12,14].

In previous work on G_2 perfect fluid models [15], in which the Einstein field equations in expansion-normalized variables take on the form of a quasi-linear hyperbolic system of autonomous partial differential equations (PDEs), it was shown that the equilibrium points of the corresponding infinite-dimensional dynamical system are represented by exact self-similar

G_2 cosmological models. Thus, it is reasonable to assume that there will be a general class of scalar-field general relativistic G_2 cosmological models with an exponential potential that will be asymptotic in the past or to the future to an exact self-similar G_2 cosmology, since the corresponding Einstein field equations again have the structure of an infinite-dimensional dynamical system in which the equilibrium points correspond to self-similar models.

In [12] scalar-field G_2 cosmologies with an exponential potential were studied. In this work the metric components of (10) were assumed to be separable in the variables t and z; *i.e.*,

$$G(t,z) = T(t)Z(z)\,, \tag{12a}$$
$$e^{F(t,z)} = e^{f(t)}e^{f_1(z)}\,, \tag{12b}$$
$$e^{p(t,z)} = Q(t)Z(z)^n\,, \tag{12c}$$

where n is a constant (note that eq. (12c) has an additional assumption that the function of z is related to the function of z in (12a)). This form of the metric has been used by several authors [16] in a different context and it allows the z-dependence of the field equations to be completely determined, leaving a set of ordinary differential equations (ODEs) for the unknown functions of t, which can be analyzed using dynamical-systems techniques. In the case studied in [14], where the function G was homogeneous, it was found that with a linear inhomogeneity (as is the case of the homogeneous Bianchi model subclass) isotropization depends solely on the parameter k of (8); when $k^2 < 2$ all solutions isotropize and homogenize, but for $k^2 > 2$ only a subclass of solutions of measure zero isotropize, although all models homogenize. For the solutions arising from (12) the analysis performed in [12] showed that most models asymptote towards an inhomogeneous class of solutions, except a subclass of massless scalar field ($V_0 = 0$) models of measure zero, for which the late-time attractor is a homogeneous Bianchi type-I model with a scalar field. The early-time attractors are inhomogeneous models which are Kasner-like in their temporal dependence.

In particular the following exact self-similar G_2 models were found to act as past or future attractors:

1) The first two equilibrium points discussed in [12] are early-time attractors (*i.e.*, sources in the dynamical system) corresponding to the cosmological model whose line element is given by

$$\mathrm{d}s^2_{\mathrm{GR}} = Dt^{C_1}z^{C_\pm}(-\mathrm{d}t^2 + \mathrm{d}z^2) + t^{1+C_2}z^{1+n}\mathrm{d}x^2 + t^{1-C_2}z^{1-n}\mathrm{d}y^2\,, \tag{13}$$

where D is an arbitrary constant,

$$m_\pm \equiv \pm\frac{\sqrt{k^2+2-2n^2}}{2n}\,, \tag{14a}$$
$$C_\pm \equiv \frac{1}{2}k\left(k \mp \sqrt{k^2+2-2n^2}\right)\,, \tag{14b}$$

$C_2 \equiv n\left(1 \pm \sqrt{8/(k^2+2)}\right)$ and $C_1 \equiv 1 + n^{-1}C_2(1+C_\pm)$ (for these solutions, the upper sign refers to one equilibrium point and the lower sign refers to the second equilibrium point). The scalar field for these two solutions is given by

$$\varphi = (m_\pm C_2 - \frac{1}{2}k)\ln t - \frac{C_\pm}{k}\ln z\,. \tag{15}$$

2) The next equilibrium point represents an important class of solutions, which are future attractors for $k^2 < 2$ (they are saddle points for $2 < k^2 < 6$ and do not exist for either $k^2 = 2$ or $k^2 > 6$). In general, they represent inhomogeneous models, but reduce to flat FRW models

in the limit $n^2 = 1$, $C_\pm = 0$. The line element describing these models is given by

$$\mathrm{d}s^2_{\mathrm{GR}} = \left[\frac{(k^2-2)}{2\sqrt{6-k^2}}t\right]^{4/(k^2-2)} \left[\frac{z^{C_\pm}}{2V_0}\left(-\mathrm{d}t^2+\mathrm{d}z^2\right) + z^{1+n}\mathrm{d}x^2 + z^{1-n}\mathrm{d}y^2\right], \tag{16}$$

and the corresponding scalar field can be written

$$\varphi = \frac{2k}{2-k^2}\ln t - \frac{C_\pm}{k}\ln z\,. \tag{17}$$

3) The final equilibrium point is a sink for $k^2 > 2$ (and a saddle for $k^2 \le 2$), and corresponds to very different models depending on the parameter k; for example, in the homogeneous limit, for $k^2 < 2$ the model is a Kantowski-Sachs model, for $k^2 = 2$ it is a flat FRW model, and for $k^2 > 2$ is a Bianchi type-III model. Defining the parameters $a^2 = \epsilon A^2(k^2-2)/(k^2+2)$, $C_3 = [2A]^{-1}[k^2A^2 \mp \epsilon a^2 k\sqrt{k^2+2-2n^2}]$ and with

$$E_z = \begin{cases} \bar{A}\cosh(az) + \bar{B}\sinh(az) & (k^2<2)\,, \\ \bar{A}z + \bar{B} & (k^2=2)\,, \\ \bar{A}\cos(az) + \bar{B}\sin(az) & (k^2>2)\,, \end{cases} \quad \epsilon = \begin{cases} +1\,, \\ 0\,, \\ -1\,, \end{cases} \tag{18}$$

where A, $\bar{A}$ and $\bar{B}$ are arbitrary constants, we can write the metric in the form

$$\mathrm{d}s^2_{\mathrm{GR}} = \frac{A^2-\epsilon a^2}{2V_0}e^{C_3 t}E_z^{C_\pm}(-\mathrm{d}t^2+\mathrm{d}z^2) + e^{\frac{A^2+n\epsilon a^2}{A}t}E_z^{1+n}\mathrm{d}x^2 + e^{\frac{A^2-n\epsilon a^2}{A}t}E_z^{1-n}\mathrm{d}y^2\,. \tag{19}$$

The scalar field in this case can be written

$$\varphi = -\frac{1}{k}\left(C_3 t + C_\pm \ln E_z\right)\,. \tag{20}$$

We shall also consider the following Bianchi type-I massless scalar-field model, corresponding to an equilibrium point which is a saddle point for all values of k, with line element

$$\mathrm{d}s^2_{\mathrm{GR}} = De^{(2+C_\pm)at}E_z^{C_\pm}(-\mathrm{d}t^2+\mathrm{d}z^2) + (e^{at}E_z)^{1+n}\mathrm{d}x^2 + (e^{at}E_z)^{1-n}\mathrm{d}y^2\,, \tag{21}$$

where $E_z \equiv A\cosh(az) + B\sinh(az)$ and a, A and B are all arbitrary constants. The massless scalar field is described by

$$\varphi = -\frac{C_\pm}{k}\left(at + \ln E_z\right)\,. \tag{22}$$

Scalar-tensor G_2 *attractors.* – Let us determine the scalar-tensor counterparts of these exact solutions.

1) Through the transformations (2), we find that the early-time attracting scalar-tensor models associated with the general relativistic solution given by eqs. (13) and (15) are described by the line element

$$\mathrm{d}s^2_{\mathrm{ST}} = \frac{t^{(k/2-m_\pm C_2)/\bar{\omega}}z^{C_\pm/k\bar{\omega}}}{\phi_0}\left(\mathrm{d}s^2_{\mathrm{GR}}\right)\,, \tag{23}$$

and by the scalar field

$$\phi = \phi_0 t^{(m_\pm C_2-k/2)/\bar{\omega}}\, z^{-C_\pm/k\bar{\omega}}\,. \tag{24}$$

We note here that the corresponding transformations are singular at $z=0$ ($t=0$ corresponds to a physical singularity). A similar situation occurs in the following cases.

2) Similarly, the late-time general relativistic attractors given by eqs. (16) and (17) are transformed into the scalar-tensor solutions described by the line interval

$$\mathrm{d}s^2_{\mathrm{ST}} = \frac{t^{-2k/[\bar{\omega}(2-k^2)]} z^{C_\pm/k\bar{\omega}}}{\phi_0} \left(\mathrm{d}s^2_{\mathrm{GR}}\right) , \tag{25}$$

and by the scalar field

$$\phi = \phi_0 \; t^{2k/[\bar{\omega}(2-k^2)]} \; z^{-C_\pm/k\bar{\omega}} . \tag{26}$$

3) The other future attractor described by eqs. (19) and (20) transforms to the scalar-tensor model described by the line interval

$$\mathrm{d}s^2_{\mathrm{ST}} = \frac{e^{\frac{C_3}{k\bar{\omega}}t} \;\; E_z^{C_\pm/k\bar{\omega}}}{\phi_0} \;\; \left(\mathrm{d}s^2_{\mathrm{GR}}\right) , \tag{27}$$

and the scalar field

$$\phi = \phi_0 e^{-\frac{C_3}{k\bar{\omega}}t} E_z^{-C_\pm/k\bar{\omega}} . \tag{28}$$

We conjecture that these self-similar G_2 scalar-tensor models play an important role in describing the asymptotic behaviour of more general scalar-tensor models. In particular, it is plausible that the exact solutions given by (23)-(24) and by (25)-(26) and (27)-(28) are attractors (past and future, respectively) for a more general class of scalar-tensor models. This conjecture may be proven by setting up the governing equations in the scalar-tensor theory as a dynamical system and determining the stability of the equilibrium points corresponding to the solutions (23)-(24), (25)-(26) and (27)-(28), or from a straightforward perturbation analysis of these solutions.

Also, the massless scalar-field model (21) and (22), corresponding to a "saddle" point, which transforms into the scalar-tensor model described by

$$\mathrm{d}s^2_{\mathrm{ST}} = \phi_0^{-1} \exp\left[\frac{aC_\pm}{k\bar{\omega}} t\right] \; E_z^{C_\pm/k\bar{\omega}} \left(\mathrm{d}s^2_{\mathrm{GR}}\right) , \tag{29}$$

and

$$\phi = \phi_0 \exp\left[\frac{-aC_\pm}{k\bar{\omega}} t\right] \; E_z^{-C_\pm/k\bar{\omega}} , \tag{30}$$

is included since this special $V = 0$ solution corresponds to a Brans-Dicke theory solution [17]. In this case the theory is formally equivalent to general relativity plus a minimally coupled massless scalar field, and we can deduce the possible asymptotic behaviour of the Brans-Dicke theory G_2 cosmological models not from the general relativistic scalar-field models with an exponential potential but from the general relativistic G_2 stiff perfect-fluid models [18].

Since the transformations (2) depend in general on both z and t, these transformations will typically be singular for a particular value of z; for example, the transformation corresponding to (24) is singular for $z = 0$. However, the transformation is well defined for $z > 0$ (for example) and scalar-tensor G_2 solutions can be obtained formally by analytic continuation. In addition, we note that this work can be generalized to scalar-tensor theories with non-constant ω [1].

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